A note on semi-direct products

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One way to think of semi-direct products is as a generalisation of the construction of dihedral groups. They crop up all the time in group theory, in number theory and in lots of other places. So here goes:

Definition 1. *Let H and N be two groups and suppose that there is an action of H on N via group automorphisms, i.e. there is a map $\phi : H \to Aut(N)$. For $h \in H$ and $n \in N$ write n^h for the image of n under the action of h, i.e. for $\phi(h)(n)$. Then the following group G is called a semi-direct product of N by H, written $N \rtimes H$:*

- *The underlying set is just the Cartesian product $N \times H$.*
- *The operation, which I call $*$ to avoid confusion with the group operations $.$ on N and H, is defined as follows:*

$$(n_1, h_1) * (n_2, h_2) = (n_1.n_2^{h_1}, h_1.h_2).$$

- *The identity element is $(1_N, 1_H)$.*
- *The inverse of (n, h) is given by $((n^{-1})^{h^{-1}}, h^{-1})$.*

It is a good way of getting a feel for this construction to check that this definition really does give a group. You should check that the operation is associative and that what I have claimed to be the inverse of an element really is the inverse.

Remarks 1.

- *It should be clear to you that due to an inherent asymmetry in the definition this group will not in general be abelian.*
- *You can check that the group $N \times \{1\}$ is normal in G.*
- *Although this aspect is not present in the notation $N \rtimes H$, the definition of G depends on the group homomorphism ϕ. Different actions of H on N will give non-isomorphic groups G. For this reason some people prefer the notation $N \times_\phi H$.*
- *If the action of H on N is trivial, in other words if $\phi(h) = id_N$ $\forall h \in H$ then we just get the direct product.*

Here are a couple of important examples:

1. The dihedral group D_n is a semi-direct product of Z_n by Z_2 where the non-trivial element of Z_2 acts on Z_n by sending everything to its inverse (just remember how you write down the dihedral group).

2. You will see (or have seen) another very similar example on the third Galois theory example sheet. In general, if some $Z_m = \langle g \rangle$ acts on some $Z_n = \langle h \rangle$ via $\phi(h)(g) = g^k$ for some natural number $k \leq m$ then the corresponding semi-direct product can be presented as

$$G = \left\{ g, h | g^m = h^n = \mathrm{id}, hgh^{-1} = g^k \right\}.$$

   This particular example is useful to have in mind when you are dealing with Kummer extensions, i.e. splitting fields of polynomials like $x^n - a$ where a is not an n-th power.

3. Here is a very funky (and extremely important) example: suppose you have any group G and suppose that H is a subgroup of the symmetric group on n letters S_n. Then consider the direct product $\underbrace{G \times \ldots \times G}_{n \text{ times}}$ and let H act on this direct product by permuting the entries just as it acts on n elements as a subgroup of S_n. Then you get a semi-direct product of $\underbrace{G \times \ldots \times G}_{n \text{ times}}$ by H. This particular construction is called the wreath product of G by H, written $G \wr H$. Note that the underlying set is $\underbrace{G \times \ldots \times G}_{n \text{ times}} \times H$ and that the operation is: multiply $(g_1, \ldots, g_n, h)$ and $(g'_1, \ldots, g'_n, h')$ term by term but before that permute the factors g'_i by the permutation h (actually by its inverse for technical reasons).

It's good fun playing with these things and one can come up with lots of concrete questions. For example, as a Christmas treat you may want to try and prove that the Sylow p-subgroup of S_{p^2} is isomorphic to $Z_p \wr Z_p$. But don't forget the pudding and the mince pies to go with this!

Enjoy your Galois theory sheets!